Springer

Journal of Mechanical Science and Technology 22 (2008) 653~661

Journal of Mechanical Science and Technology

www.springerlink.com/content/1738-494x

Modified boundary layer analysis for a mode III crack problem

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(Manuscript Received July 30, 2007; Revised January 7, 2008; Accepted January 14, 2008)

Abstract

A modified boundary layer problem of a semi-infinite crack in an elastic-perfectly plastic material under a Mode III load is analyzed. The analytic solution of elastic fields is derived by using complex function theory. It is found that the size and the shape of the plastic zone near the crack tip depend on the elastic *T*-stress given on the remote boundary. A method for determining higher order singular solutions of elastic fields is also proposed. In order to determine the higher order singular solutions of the elastic fields, Williams expansion of the solution is used. Higher order terms in the Williams expansion are obtained through simple mathematical manipulation. The coefficients of each term in the Williams expansion are also calculated numerically with the *J*-based mutual integral.

Keywords: Analytic solution; Modified boundary layer problem; Elastic T-stress; Williams expansion; J-based mutual integral

1. Introduction

An asymptotic crack problem having boundary conditions given by the leading singular term and the finite constant term of the Williams solution is called a modified boundary layer problem. A modified boundary layer problem is investigated in order to estimate the effect of the elastic *T*-stress, a finite constant term of the Williams solution, on the fracture behavior of materials. Studies on the modified boundary layer problem have been performed by several researchers. Larsson and Carlsson [1] and Rice [2] showed that the elastic *T*-stress affects the size and shape of the plastic zone around the crack tip under a small scale yielding condition. Cotterell and Rice [3] examined the stress intensity factors at the tip of a slightly curved crack and a kinked crack. They concluded that the elastic *T*-stress plays an important role in the crack stability as well as in determining the orientation of the crack path. Du and Hancock [4] numerically investigated the effect of *T*-stress on small scale yielding fields using modified boundary layer formulations. They showed that the shape of the plastic zone is significantly changed with variation of the magnitude and the sign of the elastic *T* stress. On the other hand, studies on the Mode III crack problem of an elastic plastic material have been conducted. Hult and McClintock [5] found the complete analytical solution for the simple asymptotic problem of an elastic-plastic material with a Mode III crack. Turska and Wisniewski [6] investigated uniqueness of the boundary value problem for an elastic-perfectly plastic material with a Mode III crack. Several researchers employed the *J*-based mutual integral for calculating the stress intensity factor, the elastic *T* stress, and the higher order fields around the crack tip. Cho et al. [7] utilized the *J*-based mutual integral in evaluating the stress intensity factors and the elastic *T*-stress near an interface crack tip under in-plane and anti-plane loading. Jeon and Im [8] explored the role of higher order eigenfields in elastic-plastic cracks. They introduced the *J*-based mutual integral in conjunction with finite element method for determining the complete Williams eigenfunction series for an elastic plastic crack under in-plane loading. To date, studies on modified boundary layer problems have mostly concentrated on cracking in the in-plane deformation problems. A

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DOI 10.1007/s12206-008-0113-6

survey of relevant literature reveals that few studies of the modified boundary layer problem for Mode III cracking have been reported.

The purpose of the present study is to investigate a modified boundary layer problem for a crack in an elastic-perfectly plastic material under a state of anti-plane shear deformation. In order to obtain an analytic solution for the elastic fields, an asymptotic problem with a modified boundary condition is considered. The remote boundary conditions are given by the inverse square root singular field and the elastic T stress field of the anti-plane direction. The analytic solution of the elastic field is derived by the complex function theory. It is found that the size and the shape of the plastic zone near the crack tip are strongly influenced by the elastic T-stress on the remote boundary. The plastic zone has asymmetric shape with respect to the crack surface due to the elastic T-stress. In this paper, a method for determining higher order singular solutions of the elastic fields is proposed. Higher order terms in the Williams expansion are obtained in terms of the elastic T stress by using simple mathematical manipulation. In order to ascertain the analytic solution to be correct, each term in the Williams expansion of the solution is also determined numerically on the basis of the J-based mutual integral.

2. Modified boundary layer analysis

Let us consider a modified boundary layer problem for a semi-infinite crack of Mode III in an elastic-perfectly plastic material, as shown in Fig. 1(a). Because the size of the plastic zone near the crack tip is very small compared to the crack length, the problem is assumed to be under a small scale yielding condition. The remote boundary conditions are given by the inverse square root singular term and the elastic T-stress of the Williams crack tip solution as follows:

$$\sigma_{32} + i\sigma_{31} = \frac{K_{III}}{\sqrt{2\pi z}} + iT \quad \text{as} \quad z \to \infty, \tag{1}$$

where σ_{3j} is the anti-plane shear stress. K_{III} is the stress intensity factor of Mode III, and T is the elastic T-stress in the direction of anti-plane. z is a complex variable defined as $z = x_1 + ix_2$. The crack tip is located at $x_1 = 0$ on the plane $x_2 = 0$. The crack surfaces lie on the negative x_1-direction and are free from traction:

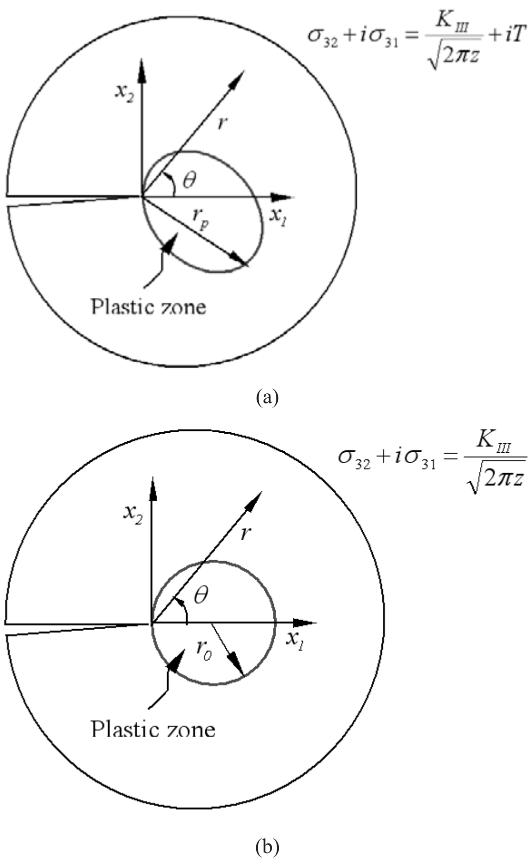

Fig. 1. Asymptotic problem of a crack in an elastic-perfectly plastic material (a) Modified boundary layer problem, (b) Simple asymptotic crack problem

$$\sigma_{32} = 0 \;\; \text{on} \;\; x_2 = 0, \;\; x_1 < 0. \tag{2}$$

Here, we review the well-known solution [9] of a simple asymptotic problem as shown in Fig. 1(b) in order to obtain the information required to analyze the modified boundary layer problem. The plastic solution of the simple asymptotic problem in the form of the Euler formula is given by

$$\sigma_{32} + i\sigma_{31} = \sigma_0 e^{-i\theta}, \text{ inside plastic zone.} \tag{3}$$

Here the cylindrical coordinates r and θ are centered at the crack tip. σ_0 is the yield stress in shear. It is noted that the boundary of the plastic zone is circular as shown in Fig. 1(b). The radius of the circular plastic zone is given by $r_0 = (K_{III}/\sigma_0)^2/2\pi$ and the origin is located at the point $(r_0, 0)$. The elastic solution of the simple asymptotic problem is also given by

$$\sigma_{32} + i\sigma_{31} = \frac{K_{III}}{\sqrt{2\pi(z - r_0)}}, \text{ outside plastic zone.} \quad (4)$$

Eqs. (1), (2), and (3) on the plastic zone boundary will be used as the boundary conditions in solving the modified boundary layer problem, and are represented in the following normalized forms:

$$\omega_2 + i\omega_1 = \frac{1}{\sqrt{\zeta}} + it \quad \text{at } D, \quad (5)$$

$$\omega_2 = 0 \quad \text{on } \overline{DEC} \text{ and } \overline{AD}, \quad (6)$$

$$\omega_2 + i\omega_1 = e^{-i\theta} \quad \text{on } \overline{CBA}, \quad (7)$$

where $\omega_j = \sigma_{3j}/\sigma_0$, $\zeta_j = x_j/r_0$, and $t = T/\sigma_0$. The points *A*, *B*, *C*, *D*, and *E* depicted in Fig. 2(a) have important meanings on the boundary of the physical domain Z_L:

A : crack tip on the lower crack surface,
B : arbitrary points on the plastic zone boundary,
C : crack tip on the upper crack surface,
D : arbitrary points at infinity,
E : point where the stress field vanishes.

Each primed and double-primed symbol on the ω-plane and the υ-plane denotes the point corresponding to the unprimed symbols *A*, *B*, *C*, *D*, and *E* on the ζ-plane in Fig. 2.

We will seek the analytic solution, satisfying the above boundary conditions, of a form given by

$$\zeta = g(\omega). \quad (8)$$

In order to obtain the analytic function, we use two complex mappings in Fig. 2. One is the mapping that transforms the domain Ω_L in the ω-plane into the domain Z_L in the ζ-plane as shown in Fig. 2. The other is the mapping that transforms the domain Ω_L in the ω-plane into the domain Y_L in the υ-plane. Let us use the function (8) as the first mapping function that changes points in the domain Ω_L into those in the domain Z_L. According to the first mapping, the boundary conditions (5), (6), and (7) on the ω-plane are given by

$$\zeta = g\left(\frac{1}{\sqrt{\zeta}} + it\right) \quad \text{as } \zeta \to \infty,$$
$$\mathrm{Im}[g(\omega)] = 0 \quad \text{on } \overline{D'E'C'} \text{ and } \overline{A'D'},$$
$$\mathrm{Im}[\omega g(\omega)] = 0 \quad \text{on } \overline{C'B'A'},$$
$$g(-i) = 0 \quad \text{at } C',$$
$$g(i) = 0 \quad \text{at } A', \quad (9)$$

where $\mathrm{Im}[\cdot]$ denotes the imaginary part of a complex variable. Let us define the second mapping function as

$$h(\upsilon) = \omega g(\omega) \quad (10)$$

with the relation of $\upsilon = (1+i\omega)^2/(1-i\omega)^2$, which sends points in the domain Ω_L to those in the domain Y_L occupying the upper half space of the υ-plane. According to the second mapping, the boundary values of the analytic function (10) on the υ-plane can be written as

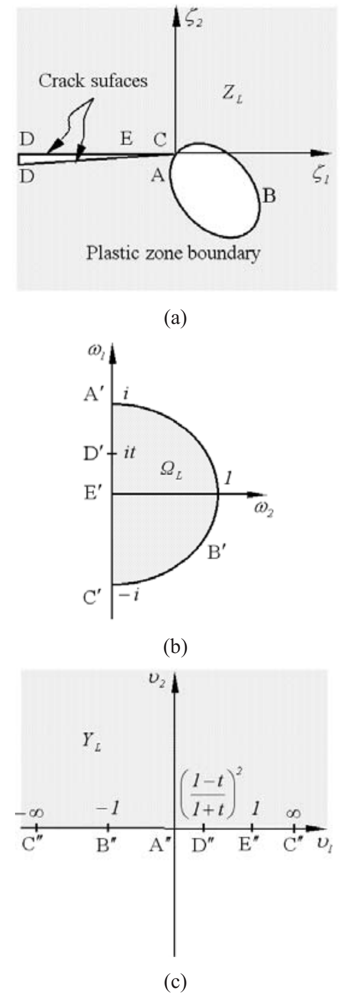

Fig. 2. Complex mappings.

$$h(\upsilon) = \left(\frac{1}{\sqrt{\zeta}} + it\right)\zeta \quad \text{as} \quad \zeta \to \infty,$$
$$\text{Re}[h(\upsilon)] = 0 \quad \text{on} \quad \overline{D''E''C''} \quad \text{and} \quad \overline{A''D''},$$
$$\text{Im}[h(\upsilon)] = 0 \quad \text{on} \quad \overline{C''B''A''},$$
$$h(\infty) = 0 \quad \text{at} \quad C'',$$
$$h(0) = 0 \quad \text{at} \quad A''. \tag{11}$$

The analytic function $h(\upsilon)$, which automatically complies with the requirements (11) on the crack surfaces, on the plastic zone boundary, and at the crack tip is represented by

$$h(\upsilon) = \frac{qi\sqrt{\upsilon}(\upsilon - 1)}{\left[\upsilon - \left(\frac{1-t}{1+t}\right)^2\right]^2}, \tag{12}$$

where $q = 4(1-t)/(1+t)^3$, which is obtained by invoking the condition at infinity in (12). Details required for the derivation procedure of q are presented in the Appendix. Finally, we can obtain the analytic solution of the modified boundary layer problem from (10)-(12) as follows:

$$g(\omega) = \frac{(1-t^2)\left(\frac{1}{\omega} + \omega\right)}{\omega\left[(1+t^2) - it\left(\frac{1}{\omega} - \omega\right)\right]^2}. \tag{13}$$

When $t=0$, Eq. (13) reduces to the well-known result obtained by Hult and McClintock [5]. This result is not identical with that of Turska and Wisniewski [6]. Their solution does not comply with the boundary condition (1) at infinity. The asymptotic fields at points outside the plastic zone from (13) can be evaluated. For the case of the dimensionless elastic T-stress $t = 0.5$, the magnitude isolines of the normalized equivalent stress are illustrated in Fig. 3. Here, $|\omega| = \sqrt{\omega_1^2 + \omega_2^2}$. It is noted that the stress vanishing point exists on the crack surface and is located at $\zeta = -(1-t^2)/t^2$. On the other hand, expressing the complex variable ζ as $\zeta = \rho e^{i\theta}$ in the form of the Euler formulae, the condition on the plastic zone boundary is given by

$$\rho_p e^{i\theta} = g(e^{-i\theta}), \tag{14}$$

where ρ_p is the dimensionless distance from the

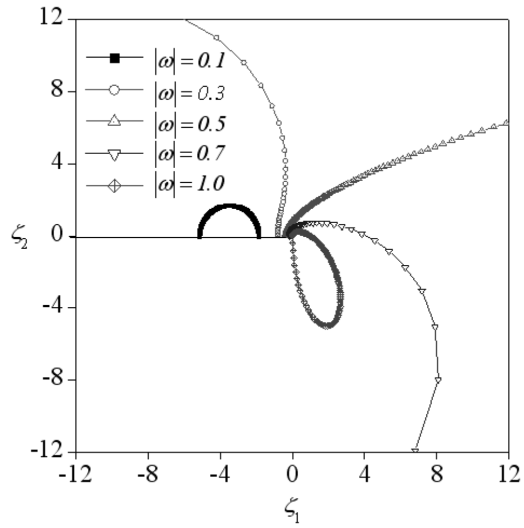

Fig. 3. Isolines of the magnitude of normalized equivalent stresses for t = 0.5.

origin to the plastic zone boundary. Substituting (14) into (13), we can obtain a formula that determines the shape of the plastic zone:

$$\rho_p = \frac{2(1-t^2)\cos\theta}{(1+t^2+2t\sin\theta)^2}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}. \tag{15}$$

The maximum dimensionless distance from the origin to the plastic zone boundary can be expressed as

$$\rho_{\max} = \frac{2(1-t^2)\cos\theta_{\max}}{(1+t^2+2t\sin\theta_{\max})^2}, \tag{16}$$

where

$$\sin\theta_{\max} = \frac{(1+t^2) - \sqrt{1+34t^2+t^4}}{4t}. \tag{17}$$

Here, $\theta_{\max}$ is the counter-clockwise angle measured from the positive x_1-axis to the angle where ρ has its maximum.

The shape of the plastic zone with variation of the dimensionless elastic T-stress is presented in Fig. 4. It is seen from Fig. 4 that the elastic T-stress has a significant effect on the shape of the plastic zone. The shape of the plastic zone is asymmetric about the crack surface in the case where the elastic T-stress is not zero.

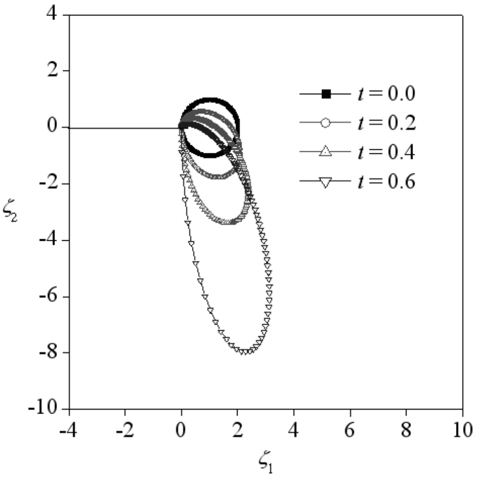

Fig. 4. Plastic zones for various normalized elastic *T*-stresses.

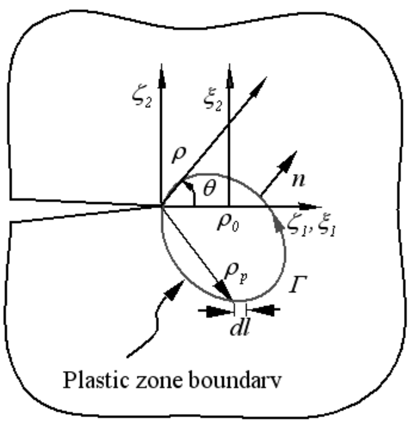

Fig. 5. Region near the crack tip in an elastic-perfectly plastic material.

3. Higher order singular solutions

We consider an outer expansion of a Williams type for the solution in order to determine higher order solutions of the elastic fields outside the plastic zone. The solution obtained in the previous section can be expanded as a Williams type series outside the plastic zone, given by

$$\omega(\xi)=\sum_{n=-\infty}^{\infty}\left(\frac{a_n}{\xi^{n+1/2}}+i\frac{b_n}{\xi^n}\right), \tag{18}$$

Here ξ is a complex variable defined as $\xi=\xi_1+i\xi_2$. The origin of the local coordinates ξ_1 and ξ_2 is situated at $(\rho_0,0)$, as depicted in Fig. 5. ξ is related to ζ as $\xi=\zeta-\rho_0$. $\rho_0=(1-t^2)/(1+t^2)^2$ is the half value of ρ_p when $\theta=0$.

Let us express (13) as

$$(\xi+\rho_0)\left[(1+t^2)\omega+it(\omega^2-1)\right]^2=(1-t^2)(1+\omega^2)\,. \tag{19}$$

Substituting (18) into (19) and equating all terms of the order ξ^α, we derive the closed forms for the coefficients a_k and b_k of the Williams expansion. At infinity, the solution (18) can be rewritten as

$$\omega(\xi)=it+\frac{1}{\xi^{1/2}}+O(\xi^{-1}) \tag{20}$$

Thus, all coefficients a_k and b_k of the Williams expansion are zero for the case of a negative integer k, and the inverse square root singular coefficients a_0 and the finite constant b_0 are $a_0=1$ and $b_0=t$, respectively. This implies that the solution (18) satisfies the boundary condition at infinity, in contrast to the result of Turska and Wisniewski [6]. We now discuss the method deriving the closed form of the coefficients a_1 and b_1 of the Williams expansion. Let us consider the three terms of the Williams expansion of the implicit solution as follows:

$$\omega(\xi)=it+\frac{1}{\xi^{1/2}}+\frac{b_1 i}{\xi}+O(\xi^{-3/2})\,. \tag{21}$$

Substitution of (21) into both sides of (19) leads to

$$\begin{aligned}&(1-t^2)^2+\frac{2i(1-t^2)(t+b_1-t^2b_1)}{\xi^{1/2}}+O(\xi^{-1})\\&=(1-t^2)^2+\frac{2it(1-t^2)}{\xi^{1/2}}+O(\xi^{-1})\end{aligned}\,. \tag{22}$$

The coefficient $b_1=0$ is obtained by comparing both sides of (22). Subsequently, let us consider the four terms of the Williams expansion of the implicit solution as follows:

$$\omega(\xi)=it+\frac{1}{\xi^{1/2}}+\frac{a_1}{\xi^{3/2}}+O(\xi^{-2})\,. \tag{23}$$

Substituting (23) into (19), we can obtain the expression given by

$$\begin{aligned}&(1-t^2)^2+\frac{2it(1-t^2)}{\xi^{1/2}}+\frac{2(1-t^2)^2a_1-t^2+\rho_0(1-t^2)^2}{\xi}\\&+O(\xi^{-3/2})=(1-t^2)^2+\frac{2it(1-t^2)}{\xi^{1/2}}+\frac{(1-t^2)}{\xi}\\&+O(\xi^{-3/2})\end{aligned}\,. \tag{24}$$

Table 1.Coefficients of the outer expansion.

k	a_k
0	1
1	$\frac{1-\rho_0(1-t^2)^2}{2(1-t^2)^2}$
2	$\frac{3-12t^2-6\rho_0(1-t^2)^2+3\rho_0^2(1-t^2)^4}{8(1-t^2)^4}$
3	$-\frac{5\left[-1+20t^2-8t^4-3\rho_0^2(1-t^2)^4+\rho_0^3(1-t^2)^6+3\rho_0(1-t^2)^2(1-4t^2)\right]}{16(1-t^2)^6}$
4	$\frac{7\left[5-280t^2+560t^4-64t^6-20\rho_0^3(1-t^2)^6+5\rho_0^4(1-t^2)^8+30\rho_0^2(1-t^2)^4(1-4t^2)-20\rho_0(1-t^2)^2(1-20t^2+8t^4)\right]}{128(1-t^2)^8}$
k	b_k
0	t
1	0
2	$-\frac{t}{(1-t^2)^3}$
3	$-\frac{2t(1-\rho_0+\rho_0t^2)}{(1-t^2)^4}$
4	$-\frac{3t\ \left[1-5t^2+t^4-2\rho_0(1-t^2)^3+\rho_0^2(1-t^2)^4\right]}{(1-t^2)^7}$

Comparison of both sides of (24) results in $a_1 = \left\{1-\rho_0(1-t^2)^2\right\}/2(1-t^2)^2$. The closed forms of the coefficients a_k and b_k can be derived by using the same procedure as (22)-(35). The closed forms of a_k and b_k for the ten terms in terms of ρ_0 and t are presented in Table 1.

Next, to evaluate the unknown coefficients a_k and b_k, we use a *J*-based mutual integral given by [10]

$$M^{(A,B)} = \int_\Gamma (w^{(A,B)}n_1 - \sigma_{ij}^A n_j u_{i,1}^B - \sigma_{ij}^B n_j u_{i,1}^A)\, dL \cdot \quad (25)$$

Here Γ is the path enclosing the crack tip that starts at a point on the lower surface and ends at a point on the upper surface of the crack, as shown in Fig. 5. *dL* is the element of the path. The superscripts *A*, *B*, and *A+B* denote the two independent equilibrium states and the superposed equilibrium elastic state, respectively. σ_{ij}, u_i, and n_i are the stress component, the displacement vector, and the unit outward normal vector, respectively. $w^{(A,B)} = \sigma_{ij}^A u_{i,j}^B = \sigma_{ij}^B u_{i,j}^A$ is the mutual energy strain energy density. The subscript comma (,) denotes a partial derivative with respect to the Cartesian coordinates. The repetition of an index in a term denotes a summation with respect to the index over its range 1 to 2. The *J*-based mutual integral refers to the interaction energy between the two independent elastic states *A* and *B*. In the case of anti-plane deformation, the *J*-based mutual reduces to

$$M^{(A,B)} = \int_\Gamma (\sigma_{3j}^A u_{3,j}^B - \sigma_{3j}^A n_j u_{3,1}^B - \sigma_{3j}^B n_j u_{3,1}^A)\, dL \cdot \quad (26)$$

Here, $u_{3,1}^A = \sigma_{31}^A/\mu$ and $u_{3,1}^B = \sigma_{31}^B/\mu$, where μ is the shear modulus. Normalizing both sides of (26) by $r_0\sigma_0^2/\mu$, we obtain the form

$$m^{(A,B)} = \int_\Gamma (\omega_j^A\omega_j^B n_1 - \omega_j^A n_j \omega_1^B - \omega_j^B n_j \omega_1^A) dl, \quad (27)$$

where $m^{(A,B)} = M^{(A,B)}(\mu/r_0\sigma_0^2)$ is the dimensionless *J*-based mutual integral. ω_j^A and ω_j^B are the anti-plane shear stresses without dimension for the equilibrium states *A* and *B*, respectively. $dl = dL/r_0$ is a dimensionless arc element. Refer to reference [10] for more details on the concept of the *J*-based mutual integral. The dimensionless *J*-based mutual integral can also be expressed in a complex form as follows:

$$m^{(A,B)} = \mathrm{Im}\left[\int_\Gamma \omega^A(\xi)\omega^B(\xi)\, d\xi\right], \quad (28)$$

where the complex functions ω^A and ω^B represent

Table 2.Coefficients of the outer expansion with various normalized elastic *T*-stresses.

t	k	J-based mutual integral		Analytic solution	
		a_k	b_k	a_k	b_k
0.1	0	1.0000000	0.1000000	1.0000000	0.1000000
	1	0.0249055	0.0000000	0.0249055	0.0000000
	2	-0.0146849	-0.1030610	-0.0146849	-0.1030610
	3	-0.0281943	-0.0081641	-0.0281943	-0.0081641
	4	-0.0023840	0.0091709	-0.0023840	0.0091709
0.3	0	1.0000000	0.3000000	1.0000000	0.3000000
	1	0.2208274	0.0000000	0.2208274	0.0000000
	2	-0.1237177	-0.3981045	-0.1237177	-0.3981045
	3	-0.5510027	-0.2651156	-0.5510027	-0.2651155
	4	-0.3435792	0.3378226	-0.3435791	0.3378225
0.5	0	1.0000000	0.5000000	1.0000000	0.5000000
	1	0.6488889	0.0000000	0.6488889	0.0000000
	2	-0.5536000	-1.1851852	-0.5536000	-1.1851852
	3	-5.7959647	-2.0227160	-5.7959647	-2.0227160
	4	-6.8102489	5.8389070	-6.8102489	5.8389070
0.7	0	1.0000000	0.7000000	1.0000000	0.7000000
	1	1.8074777	0.0000000	1.8074779	0.0000000
	2	-5.9639568	-5.2770045	-5.9639578	-5.2770051
	3	-118.9281660	-18.2696739	-118.9281932	-18.2696769
	4	33.9357985	296.5505847	33.9358431	296.5506714

the elastic fields for the equilibrium elastic state *A* and *B*, respectively. Let us take state *A* as the actual field given by (18) and set state *B* as the auxiliary field. For calculating the unknown coefficients a_k and b_k, we assume that the auxiliary field *B* is given by

$$\omega^{B_1}(\xi) = \frac{1}{2\pi}\xi^{k-1/2} \quad \text{for } a_k, \tag{29}$$

$$\omega^{B_2}(\xi) = -\frac{i}{2\pi}\xi^{k-1} \quad \text{for } b_k, \tag{30}$$

where the k is an integer. Substituting (18) and (29) into (28), we obtain the *J*-based mutual integral for the actual field *A* and the auxiliary field B_1 given by

$$m^{(A,B_1)} = \mathrm{Im}\left[\int_{\Gamma}\sum_{n=-\infty}^{\infty}\left(\frac{a_n}{\xi^{n+1/2}} + i\frac{b_n}{\xi^n}\right)\frac{1}{2\pi}\xi^{k-1/2}d\xi\right]. \tag{31}$$

Similarly, the *J*-based mutual integral for the actual field *A* and the auxiliary field B_2 is given by

$$m^{(A,B_2)} = -\mathrm{Im}\left[\int_{\Gamma}\sum_{n=-\infty}^{\infty}\left(\frac{a_n}{\xi^{n+1/2}} + i\frac{b_n}{\xi^n}\right)\frac{i}{2\pi}\xi^{k-1}d\xi\right] \tag{32}$$

Taking the plastic zone boundary as the path of integration and applying Cauchy's residue theorem to (31) and (32), we can straightforwardly obtain

$$a_k = \int_{\Gamma}(\omega_j^A\omega_j^{B_1}n_1 - \omega_j^A n_j\omega_1^{B_1} - \omega_j^{B_1}n_j\omega_1^A)dl, \tag{33}$$

$$b_k = \int_{\Gamma}(\omega_j^A\omega_j^{B_2}n_1 - \omega_j^A n_j\omega_1^{B_2} - \omega_j^{B_2}n_j\omega_1^A)dl. \tag{34}$$

The stress fields ω_j^A, $\omega_j^{B_1}$, and $\omega_j^{B_2}$ are known from (13), (29), and (30); the coefficients a_k and b_k can be calculated by integrating (33) and (34) along the plastic zone boundary. Because the coefficients are zero for the case of a negative integer *k*, the values of a_k and b_k for the case of a non-negative integer *k* are only given as the variation of the dimensionless elastic *T*-stress in Table 2. It is easily seen from (33) and (34) that the coefficients a_k and b_k obtained by using the *J*-based mutual integral depend on the shape of the plastic zone boundary. The numerical result of the coefficients a_0 and b_0 satisfies the boundary condition at infinity. This implies that the plastic zone boundary shape obtained in this paper is valid. The values of all the coefficients calculated on the basis of the *J*-based mutual integral are coincident with the values derived from the closed formulas of Table 1. The *J*-based mutual integral may be very useful in determining the solution of outer expansion for a problem of a crack with a plastic zone.

4. Concluding remarks

A modified boundary layer problem is considered in order to obtain an asymptotic solution for the elastic fields near the crack tip in elastic-perfectly plastic materials under Mode III loading. The remote boundary condition is given by the inverse square root singular field and the elastic T stress field of the anti-plane direction. The analytic solution for the stress fields is derived by using complex function theory. The shape of the plastic boundary is determined in the closed form. It is found that the size and the shape of the plastic zone near the crack tip are strongly influenced by the elastic T-stress on the remote boundary. The plastic zone has asymmetric shape with respect to the crack surface due to the elastic T-stress. A method of determining higher order singular solutions for the elastic fields is also proposed. In order to determine the higher order singular solutions for the elastic fields, a Williams type expansion of the solution is used. Higher order terms in the Williams type expansion are obtained through simple mathematical manipulation. The coefficients of each term in the Williams expansion of the solution are also calculated by using the J-based mutual integral.

Acknowledgment

This work was supported by Inha University Research Grant.

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Appendix.

A.1 Derivation of q

At infinity boundary, ω is given by

$$\omega = \frac{1}{\sqrt{\zeta}} + it \quad \text{as} \quad \zeta \to \infty . \tag{A1}$$

Defining ε as $\varepsilon = 1/\sqrt{\zeta}$, we may express (A1) as

$$\omega = \varepsilon + it . \tag{A2}$$

At infinity, the function (10) is expressed as

$$h = \frac{\varepsilon + it}{\varepsilon^2}, \quad \varepsilon \to 0 . \tag{A3}$$

The real coefficient q of the function h satisfying (A3) will be determined. The numerator $\sqrt{\upsilon}(\upsilon - 1)$ of (12) in terms of ε is given by

$$\sqrt{\upsilon}(\upsilon - 1) = -\frac{4t(1-t)}{(1+t)^2}\left[1 - i\varepsilon \frac{1-4t+t^2}{t(1+t)(1-t)}\right]. \tag{A4}$$

Here, the terms higher than the first order ε term are neglected, because their magnitude is small.

Similarly, the denominator $\left[\upsilon - \left(\frac{1-t}{1+t}\right)^2\right]$ of (12) in terms of ε is expressed as

$$\left[\upsilon - \left(\frac{1-t}{1+t}\right)^2\right]^2 = -\frac{16\varepsilon^2(1-t)^2}{(1+t)^6}\left[1 - \frac{2i\varepsilon(t-2)}{(1+t)(1-t)}\right]. \tag{A5}$$

Here, the terms higher than the first order ε term are neglected. From (A4) and (A5), Eq. (12) is given by

$$\frac{\sqrt{\upsilon}(\upsilon-1)}{\left[\upsilon-\left(\frac{1-t}{1+t}\right)^2\right]^2}=\frac{t(1+t)^3}{4\varepsilon^2(1-t)}\left[1-\frac{i\varepsilon(1-4t+t^2)}{t(1+t)(1-t)}\right]\left[1+\frac{2i\varepsilon(t-2)}{(1+t)(1-t)}\right]. \quad \text{(A6)}$$

Expanding two parenthesized expressions in the right side of (A6) and neglecting the terms higher than the first order ε term, we can obtain the following:

$$\frac{\sqrt{\upsilon}(\upsilon-1)}{\left[\upsilon-\left(\frac{1-t}{1+t}\right)^2\right]^2}=\frac{(1+t)^3(t-i\varepsilon)}{4(1-t)\varepsilon^2}. \quad \text{(A7)}$$

Substitution of (A7) into (12) results in

$$h=\frac{q(1+t)^3}{4(1-t)}\frac{\varepsilon+it}{\varepsilon^2}. \quad \text{(A8)}$$

Invoking the condition (A3), the real constant q is

$$q=\frac{4(1-t)}{(1+t)^3} \quad \text{(A9)}$$